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A gravitational model for a matter-free torsion ball

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Abstract. A classical model of a gravitational field is constructed in terms of two distinct geometries. It is formulated in terms of a purely geometric action with distinct densities in different space–time domains. An explicit static spherically symmetric solution for each geometry is found and matching properties across a world tube generated by a closed space-like submanifold are discussed in terms of the associated second fundamental forms. The interior geometry has bounded curvature and torsion. The exterior region has a Schwarzschild geometry. The model simulates distinct phases of the gravitational field and is motivated by E Cartan's analogy of torsion effects with dislocation phenomena in continuum mechanics.

1. Introduction

One of the most striking features of Einstein's theory of gravity is its prediction of certain space–time domains that are unobservable to certain observers and the existence of singularities (Penrose 1969, Hawking and Ellis 1973). These features are found to hold in a large class of field configurations that seem relevant in this cosmos. It is sometimes asserted that quantum gravity will modify these aspects of classical gravity and smooth out space–time singularities but there is no viable demonstration of this at present. It may be argued that by modifying the classical action for gravity it is possible to discover field configurations that are indistinguishable from the Schwarzschild geometry at large distances from a centre (or axis) of symmetry but nevertheless have a non-singular geometry. In many ways this is the only escape route from black-hole singularities with static spherical symmetry since by Birkhoff's theorem there are no alternative vacuum Einstein spaces.

A possible modification is to assume collapsing matter with enough intrinsic spin to activate a strong torsion phase for the geometry within the matter. It has been proposed that the presence of such a torsion can modify the nature of the geometry enough to arrest the formation of a black hole from an intrinsically spinning star (Trautman 1973). However, a curvature singularity may not always arise from collapsed matter (the Schwarzschild geometry is a static vacuum solution with a particular boundary condition) in which case an alternative mechanism must be sought. One approach is to seek a mechanism of torsion generation that does not depend explicitly on the equation of state of spinning matter. That is, we require an exact solution to a theory in which the connection describing the gravitational field generates torsion without the intervening effects of matter.

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The modifications to Einstein's theory which exist in the literature may be regarded as mathematical models whose properties might suggest viable tests in the real world. Many modifications are formulated in terms of a gauge group and a set of associated gauge fields (see also Moffat and Taylor 1980).

In a series of articles (Benn *et al* 1980a, b, c) we have demonstrated that by regarding gravity in terms of an SL(2, C) gauge theory of space–time frames, a natural and simple extension of Einstein's original theory is obtained. Within this formalism we ask whether it is possible to construct a matter-free action purely in terms of geometrical variables that possesses exact singularity-free solutions approaching the Schwarzschild geometry in some coordinate system.

Our search for an appropriate dynamics is motivated by Cartan's (1922) analogy of torsion to continuum dislocations (e.g. Landau and Lifshitz 1970). If one imagines a dense lattice of affine frames over space–time then each lattice point requires for its specification six orientation coordinates and four origin coordinates with respect to the additional space–time event coordinates. The origin of each frame must be visualised in the tangent space of the space–time manifold. In a space without torsion the origin of all the affine frames are correlated rather like the bonds of a perfect atomic lattice. In a space with torsion the affine frame field is such that after returning from a smooth detour in the crystal one does not obtain a unique origin for the affine frame (just as the direction of a vector is not preserved after performing a circuit in a space with intrinsic curvature). This is reminiscent of the definition of the Burgers' vector for dislocations in an atomic lattice. Although the analogy is far from perfect it is not the first time that geometrical descriptions of gravity have been compared with other systems in physics (Sakharov 1967). Now atomic lattice dislocations have their own dynamics and can sometimes coalesce to form dislocation boundaries. The dislocation density can be very different on either side of the boundary. If space–time torsion is to be considered in terms of dislocated affine frames then a natural approach might be to seek an action for some bounded space-like region that describes a torsion phase and a distinct action for an exterior torsion-free domain. We then restrict ourselves to spherically symmetric static field configurations and seek exact solutions that have a continuous metric across the boundary.

2. The model

In an earlier article (Benn *et al* 1980a), whose notation and conventions we follow here, we have argued how a classical gravitational soliton in space–time might arise in a theory with the appropriate action. In that paper we relied on the existence of a 'gravitationally charged' field to hold the soliton together. We remarked that it might be possible to view this field as a manifestation of the local contortion of space–time.

We are guided in the following by some of the mechanisms discussed in that paper. However, we shall suppose that the geometry of space–time be partitioned into two distinct phases separated by a dynamical boundary. We may give the boundary an intrinsic formulation as the image of a 3-chain on space–time with a closed space-like 2-surface. The dynamics of free relativistic membranes has been investigated elsewhere together with certain supersymmetric extensions (Collins and Tucker 1976, Howe and Tucker 1978). In the model to be constructed we completely neglect this aspect of the problem to the extent that we seek two static spherically symmetric gravitational phases separated by a static spherically symmetric interface that partitions the world tube of a three-dimensional hypersurface from the rest of space–time.

The interface will be expected to sustain a discontinuous SL(2, C) curvature although we demand that the metric components be continuous across it in some basis. (It might in fact be found possible to arrange a C^1 metric across the interface thus ensuring smooth Christoffel geodesics linking the two geometries.) We shall seek an interior geometry with torsion but a torsion-free Einstein geometry on the exterior domain. Then the full SL(2, C) curvature of the full SL(2, C) connection will be discontinuous at the interface. This will set up stresses that must be sustained by our postulated boundary. We shall assume that the action of the 3-chain generates a dynamics that enables the two geometries to remain stable. (This is analogous to requiring the stresses generated by the electric field of a hollow electrically charged conducting sphere to remain in equilibrium with the elastic forces of the conductor.)

We know that the only torsion-free Einstein space with static spherical symmetry has an exterior Schwarzschild metric and we adopt for the exterior geometry the Einstein–Hilbert action

$$\mathcal{S}_{\mathrm{I}}[e, \omega] = G \int_{M_{\mathrm{I}}} \mathrm{SIm}\{\hat{R} \wedge e \wedge \bar{e}\} \tag{1}$$

regarded as a functional of the anti-Hermitian coframe e and the SL(2, C) connection 1-form $\hat{\omega}$ which are defined below. $\hat{R}$ denotes the corresponding SL(2, C) curvature. SIm stands for the scalar-imaginary part of the complex quaternion in the parentheses. G is Newton's gravitational constant. In view of the earlier comments on gravitational solitons we are motivated to adopt a different action $\mathcal{S}_{\mathrm{II}}$ involving both curvature and torsion for the interior domain M_{II}. Non-Einsteinian actions have been much studied recently. In particular, in a number of distinct actions involving quadratic curvatures apparently similar geometries containing torsion in the absence of spinning matter have been found (Benn *et al* 1980c, Baekler 1981). However, in the approach here, we are constrained by requiring a model to link an exact field configuration to that of the usual exterior Schwarschild geometry.

To construct an appropriate action for this purpose we introduce the contortion 1-form $\hat{K}$. This is defined in terms of the full SL(2, C) connection $\hat{\omega}$ by

$$\hat{K} = \hat{\omega} - \hat{\Gamma} \tag{2}$$

where $\hat{\Gamma}$ is the torsion-free (Christoffel) connection defined in terms of the anti-Hermitian coframe e by†

$$de + 2\mathcal{A}(\hat{\Gamma} \wedge e) = 0. \tag{3}$$

In order to maintain the transformation

$$\hat{\omega} \to Q\hat{\omega}\bar{Q} + Q\mathrm{d}\bar{Q} \tag{4}$$

under a change of gauge section generated by SL(2, C) in terms of the complex unit norm quaternion Q, the contortion form has adjoint transformation properties

$$\hat{K} \to Q\hat{K}\bar{Q}. \tag{5}$$

† In terms of orthonormal basis 1-forms e^a, $a = 0, 1, 2, 3$, the anti-Hermitian coframe is given by $e = \mathrm{i}e^0 + \Sigma_{k=1}^3 e^k \hat{e}_k$ where $\hat{e}_k$ are three algebraic quantities satisfying $\hat{e}_k\hat{e}_l = -\delta_{kl} + \hat{e}_j$, (klj) cyclic. $\mathcal{A}$ denotes the anti-Hermitian, $\mathcal{H}$ the Hermitian part of any quaternionic form.

In terms of $\hat{K}$ the torsion 2-form T is given by

$$T = 2\mathscr{A}(\hat{K} \wedge e). \tag{6}$$

Since the full curvature $\hat{R}$ is defined as

$$\hat{R} = \mathrm{d}\hat{\omega} + \hat{\omega} \wedge \hat{\omega} \tag{7}$$

we can write this in terms of the Christoffel curvature

$$\hat{R}_\Gamma = \mathrm{d}\hat{\Gamma} + \hat{\Gamma} \wedge \hat{\Gamma} \tag{8}$$

as

$$\hat{R} = \hat{R}_\Gamma + \mathrm{d}\hat{K} + 2V(\hat{\Gamma} \wedge \hat{K}) + \hat{K} \wedge \hat{K}. \tag{9}$$

The natural quadratic curvature 4-form is $\hat{R} \wedge *\hat{R}$ where $*$ is the Hodge dual operator with respect to the space–time metric. In the absence of torsion, i.e. $(T = 0)$, this is simply $\hat{R}_\Gamma \wedge *\hat{R}_\Gamma$. It should be observed at this point that in the presence of torsion, Re$S\{\hat{R}_\Gamma \wedge *\hat{R}_\Gamma\}$ generates an SL(2, C) gauge covariant theory distinct from that described by the action Re $S\{\hat{R} \wedge *\hat{R}\}$. This is because (3) and (6) can be explicitly solved for $\hat{\Gamma}$ and $\hat{K}$ in terms of the gauge covariant forms T and e

$$\hat{\Gamma} = \tfrac{1}{8}(\bar{i}_X i_X)S(\mathrm{d}e \wedge \bar{e}) + \tfrac{1}{2}V(\bar{i}_X\, \mathrm{d}\bar{e}) \tag{10}$$

$$\hat{K} = -\tfrac{1}{8}(\bar{i}_X i_X)S(T \wedge \bar{e}) - \tfrac{1}{2}V(\bar{i}_X \bar{T}) \tag{11}$$

where

$$i_X = i i_0 + \sum_{k=1}^{3} i_k \hat{e}_k \qquad \text{and} \qquad i_a(e^b) = \delta_a^b \qquad a, b = 0, 1, 2, 3$$

so that from (9), Re $S\{\hat{R}_\Gamma \wedge *\hat{R}_\Gamma\}$ may be expressed solely in terms of $\hat{\omega}$ and e. These can be varied independently to generate SL(2, C) covariant equations. The component form of such an action is given in the Appendix. In terms of the contortion our model is described by the complete SL(2, C) invariant action†

$$\mathscr{S}[e, \hat{\omega}, \pi] = -G\ \mathrm{Im} \int_{M_\mathrm{I}} S(\hat{R} \wedge e \wedge \bar{e}) + \alpha\ \mathrm{Re} \int_{M_\mathrm{II}} S(\hat{R}_\Gamma \wedge *\hat{R}_\Gamma + D_\Gamma\hat{K} \wedge *D_\Gamma\hat{K}) + \int_{C_3} \pi \tag{12}$$

where α is a real constant and the 3-chain C_3 bounds M_II and π is an appropriate 3-form action for the dynamics of C_3. We define $D_\Gamma\hat{K} \equiv \mathrm{d}\hat{K} + 2V(\hat{\Gamma} \wedge \hat{K})$. The field equations from independent variations of e and ω in the region M_I and M_II are rather complex. In keeping with our intention to examine static solutions we shall ignore the contribution of π to these equations and simply replace its effects by the boundary conditions discussed above.

3. Static solution

The geometry of M_I is determined by the Einstein–Cartan system of equations

$$\delta\hat{\omega}: \qquad D(e \wedge \bar{e}) = 0 \tag{13}$$

$$\delta e: \qquad \mathscr{H}(R \wedge e) = 0. \tag{14}$$

† The kinetic term for the contortion in the second integral is suggested by a formal analogy with the Yang–Mills–Higgs systems.

Equation (13) implies $T=0$ in this region and we have an Einstein space with the unique spherically symmetric Schwarzschild solution for $r \geq 2m$

$$e^{\mathrm{I}} = \mathrm{i}\left(1-\frac{2m}{r}\right)^{1/2} \mathrm{d}t + \frac{N\,\mathrm{d}r}{(1-2m/r)^{1/2}} + r\,\mathrm{d}N \tag{15}$$

$$\hat{\Gamma}^{\mathrm{I}} = -\frac{\mathrm{i}m}{2r^2} N\,\mathrm{d}t + \frac{1}{2}\left[\left(1-\frac{2m}{r}\right)^{1/2} - 1\right] N\,\mathrm{d}N \tag{16}$$

where in terms of R^3 chart coordinates (r, θ, ϕ), N is the unit q vector

$$N = \hat{e}_3 \cos\theta + \sin\theta \sin\phi \hat{e}_2 + \sin\theta \cos\phi \hat{e}_1.$$

In this gauge the SL(2, C) curvature 2-form in region I is

$$\hat{R}^{\mathrm{I}} = -\frac{\mathrm{i}m}{2r^2}\left(1-\frac{2m}{r}\right)^{1/2} \mathrm{d}N \wedge \mathrm{d}t + \frac{\mathrm{i}m}{r^3} N\,\mathrm{d}r \wedge \mathrm{d}t + \frac{m}{2r^2}\left(1-\frac{2m}{r}\right)^{-1/2} \mathrm{d}r \wedge N\,\mathrm{d}N - \frac{m}{2r}\mathrm{d}N \wedge \mathrm{d}N \tag{17}$$

and the gauge invariant quadratic curvature 4-form

$$-\mathrm{Re}\, S(\hat{R}^{\mathrm{I}} \wedge *\hat{R}^{\mathrm{I}}) = \frac{3m^2}{2r^6} *1 \tag{18}$$

where $*1$ is the volume element on M_{I}.

Since

$$\mathrm{Re}\int_{M_{\mathrm{II}}} S(\{\hat{R}_\Gamma \pm \mathrm{i} * D_\Gamma \hat{K}\} \wedge *\{\hat{R}_\Gamma \pm \mathrm{i} * D_\Gamma \hat{K}\}) = \mathrm{Re}\int_{M_{\mathrm{II}}} S(\hat{R}_\Gamma * \hat{R}_\Gamma + D_\Gamma \hat{K} * D_\Gamma \hat{K}) \pm \int_{M_{\mathrm{II}}} \mathrm{d}(\mathrm{Re}\, S\{2\mathrm{i}\hat{R}_\Gamma \wedge \hat{K}\}) \tag{19}$$

we see that as far as the interior field equations in region M_{II} are concerned they follow from the action

$$\mathcal{S}_{\mathrm{II}} = \alpha\, \mathrm{Re}\int_{M_{\mathrm{II}}} S(\hat{\rho} \wedge *\hat{\rho}) \tag{20}$$

where $\hat{\rho} = \hat{R}_\Gamma \pm \mathrm{i} * D_\Gamma \hat{K}$. The extra term can be transformed to a 3-form over C_3. The interior field equations are

$$\delta\hat{\omega}: \qquad V\left(\hat{\rho} \wedge * \frac{\delta\hat{\rho}}{\delta\omega}\right) = 0 \tag{21}$$

$$\delta e: \qquad \mathcal{H}\hat{\rho} \wedge i_X * \hat{\rho} - *\hat{\rho} \wedge i_X\hat{\rho}) = 0. \tag{22}$$

To solve the field equations in region M_{II} we adopt solutions that arise from $\hat{\rho} = 0$ or

$$i * \hat{R}^{\mathrm{II}}_\Gamma = \eta D_\Gamma \hat{K} \tag{23}$$

in terms of a polarity parameter $\eta = \pm 1$†.

† It may be of some interest to note here that the integrability conditions

$$D_\Gamma * \hat{R}_\Gamma = *D\hat{K} \wedge \hat{K} - \hat{K} \wedge *D\hat{K} \qquad D_\Gamma * D_\Gamma \hat{K} = 0$$

of equation (23) are structurally similar to the Yang–Mills–Higgs field equations.

The most general spherically symmetric static forms for e and $\hat{K}$ are (up to gauge equivalence)

$$e = \mathrm{i}h_0(r) + h_1(r)N\,\mathrm{d}r + h_2(r)\,\mathrm{d}N \tag{24}$$

$$\hat{K} = \mathrm{i}f_0(r)N\,\mathrm{d}t + f_1(r)N\,\mathrm{d}r + f_2(r)\,\mathrm{d}N + f_3(r)N\,\mathrm{d}N \tag{25}$$

where h_0, h_1 and h_2 are real functions of r but the f_i, $i = 0, 1, 2, 3$, may be complex. The complex q-vector valued 2-form equation (23) now yields the following set of coupled differential equations:

$$(h_2^2/2h_0h_1)p_0' = \eta f_3(1+2p_1) \tag{26}$$

$$p_0(2p_1+1)h_1/h_0 = \eta f_3' \tag{27}$$

$$(h_0/h_1)p_1' = \eta(f_0 + 2p_0f_3 + 2p_1f_0) \tag{28}$$

$$(2h_0h_1/h_2^2)p_1(1+p_1) = \eta f_0' \tag{29}$$

$$p_0f_2 = 0 \tag{30}$$

$$f_2' = f_1(2p_1+1) \tag{31}$$

where $p_0 = -h_0'/2h_1$ and $p_1 = (h_2' - h_1)/2h_1$. *The* $'$ denotes $\mathrm{d}/\mathrm{d}r$.

A particular solution to this set that satisfies the constraints we desire is

$$f_0 = -\frac{\eta\lambda r}{2r_0^2} + a \tag{32}$$

$$h_0^2 = \lambda^2/h_1^2 = b_1r + b_0 \tag{33}$$

$$h_2 = r_0 \tag{34}$$

where a, b_0, b_1, λ, r_0 are real constants and all other functions are zero.

In order to fix these constants we match geometries as far as possible across the interface between M_{I} and M_{II} (Synge 1966, Israel 1965)†. It transpires that the most natural matching occurs on the submanifold $r = 2m$ for all t in a chart that includes the image of C_3. The constant a is fixed to be $\eta\lambda m/r_0^2$ so that $f_0(2m) = 0$. Since

$$\hat{\omega}^{\mathrm{II}} = \mathrm{i}(b_1/4\lambda + f_0(r))N\,\mathrm{d}t - \tfrac{1}{2}N\,\mathrm{d}N \tag{35}$$

then

$$\hat{\omega}^{\mathrm{II}}|_{r=2m} = \hat{\omega}^{\mathrm{I}}|_{r=2m}. \tag{36}$$

Furthermore if we set $b_1 = \lambda/2m$ and $b_0 = -\lambda$ then

$$e^{\mathrm{I}}|_{r=2m} = e^{\mathrm{II}}|_{r=2m}. \tag{37}$$

We adjust the signature convention of the interior Lorentzian metric by setting $\lambda = -\mu^2$ $(m > 0)$ so that the final choice of interior geometry is given in terms of the (t, r, θ, ϕ) chart as: $(r \leqslant 2m)$.

$$e^{\mathrm{II}} = \mathrm{i}\mu(1 - r/2m)^{1/2}\,\mathrm{d}t - \mu(1-r/2m)^{-1/2}N\,\mathrm{d}r + 2m\,\mathrm{d}N \tag{38}$$

$$g^{\mathrm{II}} = -\mu^2(1-r/2m)\,\mathrm{d}t\otimes\mathrm{d}t + \mu^2\frac{\mathrm{d}r\otimes\mathrm{d}r}{(1-r/2m)} + 4m^2\{\sin^2\theta\,\mathrm{d}\phi\otimes\mathrm{d}\phi + \mathrm{d}\theta\otimes\mathrm{d}\theta\} \tag{39}$$

$$*1 = 2\mathrm{m}^2\mu^2\,\mathrm{d}r\wedge\mathrm{d}t\wedge N\,\mathrm{d}N\wedge\mathrm{d}N \tag{40}$$

† We have learnt that the junction conditions with torsion are also discussed by Arkuszewski *et al* (1974).

$$\hat{\omega}^{\mathrm{II}}=\mathrm{i}\left[-\frac{1}{8m}+\frac{\eta\mu^2}{4m}\left(\frac{r}{2m}-1\right)\right]N\,\mathrm{d}t-\frac{1}{2}N\,\mathrm{d}N \tag{41}$$

$$\hat{R}^{\mathrm{II}}=\frac{\mathrm{i}\eta\mu^2}{8m^2}N\,\mathrm{d}r\wedge\mathrm{d}t-\frac{1}{4}\mathrm{d}N\wedge\mathrm{d}N \tag{42}$$

$$T^{\mathrm{II}}=(\mathrm{i}\eta\mu^3/2m)(1-r/2m)^{1/2}\,\mathrm{d}r\wedge\mathrm{d}t \tag{43}$$

$$T^{\mathrm{II}}\wedge *\bar{T}^{\mathrm{II}}=\tfrac{1}{4}m^2(1-r/2m)*1 \tag{44}$$

$$\hat{R}^{\mathrm{II}}\wedge *\hat{R}^{\mathrm{II}}=-\tfrac{1}{32}m^{-4}*1. \tag{45}$$

It will be noticed that the gauge invariant 4-form $T^{\mathrm{II}}\wedge *\bar{T}^{\mathrm{II}}$ vanishes on the interface $r=2m$ and the curvature invariant $\hat{R}^{\mathrm{II}}\wedge *\hat{\mathrm{R}}^{\mathrm{II}}$ is non-singular. For comparison the external geometry is specified by (15), (16) and (17) together with

$$\begin{aligned} g^{\mathrm{I}}&=-(1-2m/r)\,\mathrm{d}t\otimes\mathrm{d}t+\frac{\mathrm{d}r\otimes\mathrm{d}r}{(1-2m/r)}+r^2\{\sin^2\theta\,\mathrm{d}\phi\otimes\mathrm{d}\phi+\mathrm{d}\theta\otimes\mathrm{d}\theta\}\\ T^{\mathrm{I}}&=0. \end{aligned} \tag{46}$$

Although the metrics and connections are continuous at the interface there is a curious frame reflection that occurs. The space or time frame parity conventions are opposite across the interface if a common coordinate chart is used to establish these conventions.

Another curious feature of the solution under discussion is the double duality property:

$$*\hat{R}^{\mathrm{I}}=\mathrm{i}\hat{R}^{\mathrm{I}} \qquad \text{in } M_{\mathrm{I}} \tag{47}$$

$$*\hat{R}^{\mathrm{II}}=-\mathrm{i}\eta\hat{R}^{\mathrm{II}} \qquad \text{in } M_{\mathrm{II}}. \tag{48}$$

Property (47) is to be expected since all Einstein spaces have double dual curvatures. Property (48) is surprising since the action for M_{II} is non-Einsteinian. It implies that for a negative polarity torsion ($\eta=-1$)

$$\mathcal{H}(\hat{R}^{\mathrm{II}}\wedge e^{\mathrm{II}})=-\frac{\mathrm{i}}{8m^2}*e^{\mathrm{II}}. \tag{49}$$

This is recognised as an Einstein–Cartan equation with a 'cosmological term' and indicates that the $\eta=-1$ geometry also arises as a double dual solution of the field equation

$$\delta\Lambda^{\mathrm{II}}/\delta e=0$$

arising from an interior action

$$\int_{M_{\mathrm{II}}}\Lambda^{\mathrm{II}}=-2\,\mathrm{Im}\int_{M_{\mathrm{II}}}S\left(\hat{R}\wedge e\wedge\bar{e}+\frac{1}{2m^2}*1\right). \tag{50}$$

The volume 4-form $*1$ seems to be simulating an effective 'pressure' as a source for the interior geometry. However, the action (12) is preferred to this one since the latter would predict zero torsion from

$$\delta\Lambda^{\mathrm{II}}/\delta\hat{\omega}=0.$$

Furthermore the $\eta=+1$ polarity solution does not satisfy (49).

The curvature 2-forms can be pulled back to the 2-dimensional domain $r=2m$, $t=\text{constant}$ and we find

$$\beta^*\hat{R}^{\text{II}}=\beta^*\hat{R}^{\text{I}}=-\tfrac{1}{4}\,\mathrm{d}N\wedge\mathrm{d}N \tag{51}$$

where β^* denotes the pull back. However, the orthonormal components are not continuous across the interface $r=2m$ that partitions the three-dimensional space-like hypersurface $t=\text{constant}$. In order to study the intrinsic nature of the interface we examine the second fundamental forms of the surfaces $M^{(3)}$ and $M^{(2)}$ whose tangent spaces are spanned by X_1, X_2, X_3 and X_2, X_3 respectively where X_0, X_1, X_2, X_3 form an orthonormal frame with respect to the generic static spherically symmetric metric

$$g=-h_0^2(r)\,\mathrm{d}t\otimes\mathrm{d}t+h_1^2(r)\,\mathrm{d}r\otimes\mathrm{d}r+h_2^2(r)\{\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2\theta\,\mathrm{d}\phi\otimes\mathrm{d}\phi\} \tag{52}$$

i.e. $g(X_i, X_j)=\eta_{ij}=\text{diag}(-1,1,1,1)$. Thus

$$X_0=\frac{1}{h_0}\frac{\partial}{\partial t},\qquad X_1=\frac{1}{h_1}\frac{\partial}{\partial r},\qquad X_2=\frac{1}{h_2}\frac{\partial}{\partial\theta},\qquad X_3=\frac{1}{h_2\sin\theta}\frac{\partial}{\partial\phi}. \tag{53}$$

For the three-dimensional submanifold $M^{(3)}$, the second fundamental form $H^{(0)}$ is the symmetric (2, 0) tensor defined by (Kobayashi and Nomizu 1969)

$$(\nabla_X Y)(e^0)=H^{(0)}(X,Y)\qquad\forall X, Y\in T(M^{(3)}) \tag{54}$$

where ∇ is the torsion-free connection on M calculated from g. In terms of the connection components Γ^i_{jk} defined by

$$\nabla_{X_i}X^j=\Gamma^k_{ij}X_k\qquad i,j,k=0,1,2,3 \tag{55}$$

$$H^{(0)}=\sum_{m,k=1}^{3}\Gamma^0_{mk}e^m\otimes e^k \tag{56}$$

where $e^i(X_j)=\delta^i_j$. For the two-dimensional submanifold $M^{(2)}$ the second fundamental forms $\tilde{H}^{(0)}$ and $\tilde{H}^{(1)}$ are defined by

$$(\nabla_X Y)(e^0)=\tilde{H}^{(0)}(X,Y) \tag{57}$$

$$(\nabla_X Y)(e^1)=\tilde{H}^{(1)}(X,Y)\qquad\forall X, Y\in T(M^{(2)}). \tag{58}$$

Thus

$$\tilde{H}^0=\sum_{m,k=1}^{2}\Gamma^0_{mk}e^m\otimes e^k \tag{59}$$

$$\tilde{H}^{(1)}=\sum_{m,k=1}^{2}\Gamma^1_{mk}e^m\otimes e^k. \tag{60}$$

The Γ^k_{ij} components with respect to this orthonormal basis are calculated from

$$g(\nabla_{X_i}X_j, X_k)=g([X_i,X_j],X_k)+g([X_k,X_i],X_j)+g([X_k,X_j],X_i). \tag{61}$$

The non-vanishing commutators needed are:

$$[X_0,X_1]=(h_0'/h_1h_0)X_0 \tag{62}$$

$$[X_1,X_2]=-(h_2'/h_1h_2)X_2 \tag{63}$$

$$[X_1,X_3]=-(h_2'/h_1h_2)X_3 \tag{64}$$

$$[X_2,X_3]=-(\cot\theta/h_2)X_3 \tag{65}$$

and we readily calculate

$$H^{(0)} = \tilde{H}^{(0)} = 0 \tag{66}$$

$$\tilde{H}^{(1)} = (h_2'/h_1h_2)\{e^2 \otimes e^2 + e^3 \otimes e^3\}. \tag{67}$$

Computing $\tilde{H}^{(1)}$ with the two metrics g^{I} and g^{II} we find that they agree at $r = 2m$. According to a statement in Misner and Sharp (1964), if the second fundamental forms of g^{I} and g^{II} associated with the three-dimensional space-like hypersurface agree on the interface then there exists a chart in which the components of g^{I} and g^{II} and their first derivatives are continuous at $r = 2m$. This would imply continuity of the Christoffel geodesics across the interface.

4. Summary

We have presented a classical model of a matter-free gravitational field that exists in two distinct phases. One of the phases is considered to be vacuum Einstein. We have attempted to motivate the description of the torsion phase by analogy with the physics of other gauge systems. For the exact spherically symmetric static solution discussed here, the interior torsion phase is fixed by its matching properties with an exterior Schwarzschild geometry. Although it is not *a priori* clear what physical conditions should determine the jump conditions across the interface between the two distinct gravitational phases in general, we should note that the complete solution found here meets the same requirements as those usually adopted for other gravitational junctions.

One conclusion we can draw from this exercise is that it may be possible to prevent the formation of space–time singularities by a gravitational 'phase transition' involving a dynamical torsion phase. If a stability analysis confirms that the solution discussed here is not unphysical we would suggest this model as a mechanism for a 'cosmic censor'.

Our solution is certainly not unique. However, we feel that further investigations should first attempt to understand the time-dependent approach to the geometries adopted in this paper before dwelling on uniqueness questions. A proper stability analysis must take into account the dynamics of the boundary and consequently one should not ignore either the action of the interface or the exact 4-form that has been cast onto the associated 3-chain. Finally it should be noted that matter couplings to the geometry in the presence of the torsion phase may be essentially distinct from those found in Einstein's theory.

Appendix

For those who are more familiar with the component formulation of gauge theories of gravitation we present here a component formulation of our model.

We start from the orthonormal basis 1-forms e^a and linear connection 1-forms $\omega^a_{\ b}$. The indices $a, b, \ldots,$ are raised and lowered by the Minkowski metric $\eta_{ab} = (-+++)$. The metric compatibility of the linear connection is ensured by $\omega_{ab} = -\omega_{ba}$. The metric of the space–time is $g = \eta_{ab} e^a \otimes e^b$. The contortion 1-forms $K^a_{\ b}$ are defined by

$$K^a_{\ b} = \omega^a_{\ b} - \Gamma^a_{\ b} \tag{A1}$$

where the structure equations

$$\mathrm{d}\,e^{a}+\Gamma^{a}_{b}\wedge e^{b}=0 \tag{A2}$$

determine uniquely the (Christoffel) connection 1-forms Γ^{a}_{b}. The torsion 2-form

$$T^{a}=K^{a}_{b}\wedge e^{b} \tag{A3}$$

and the curvature 2-form

$$R^{a}_{b}=\mathrm{d}\omega^{a}_{b}+\omega^{a}_{c}\wedge\omega^{c}_{b}. \tag{A4}$$

We define

$$(DK)^{a}_{b}=\mathrm{d}K^{a}_{b}+\omega^{a}_{c}\wedge K^{c}_{b}+K^{a}_{c}\wedge\omega^{c}_{b} \tag{A5}$$

and

$$(D_{\Gamma}K)^{a}_{b}=\mathrm{d}K^{a}_{b}+\Gamma^{a}_{c}\wedge K^{c}_{b}+K^{a}_{c}\wedge\Gamma^{c}_{b}. \tag{A6}$$

It follows from above that

$$(D_{\Gamma}K)^{a}_{b}=(DK)^{a}_{b}-2K^{a}_{c}\wedge K^{c}_{b}. \tag{A7}$$

Furthermore, defining the (Christoffel) curvature 2-forms

$$R_{\Gamma}{}^{a}_{b}=\mathrm{d}\Gamma^{a}_{b}+\Gamma^{a}_{c}\wedge\Gamma^{c}_{b} \tag{A8}$$

we find

$$R^{a}_{\Gamma b}=R^{a}_{b}-(DK)^{a}_{b}+K^{a}_{c}\wedge K^{c}_{b}. \tag{A9}$$

The total action of our model is

$$\mathscr{S}(e,\omega,\pi)=\frac{G}{2}\int_{M_{\mathrm{I}}}R_{ab}\wedge *(e^{a}\wedge e^{b})$$

$$+\frac{\alpha}{4}\int_{M_{\mathrm{II}}}\{R^{ab}_{\Gamma}\wedge *R_{\Gamma ab}+(D_{\Gamma}K)^{ab}\wedge *(D_{\Gamma}K)_{ab}\}+\int_{C_{3}}\pi. \tag{A10}$$

It is possible to re-express the second integral above solely in terms of e and ω and their derivatives. In order to do this we first rewrite equation (A3) in component form as

$$T_{ab,c}=-K_{a,bc}+K_{b,ac}. \tag{A11}$$

We use this relation to solve for the contortion components in terms of the torsion components:

$$2K_{a,bc}=T_{ab,c}-T_{ca,b}+T_{bc,a}. \tag{A12}$$

Substituting into (A7) and (A9) we rewrite these equations in the component language:

$$2\underset{(\Gamma)}{\nabla}_{[c}K_{d]},{}^{a}{}_{b}=\underset{(\omega)}{\nabla}_{[c}T_{d]b},{}^{a}-\underset{(\omega)}{\nabla}_{[c}T_{d]}{}^{a},{}_{b}-\underset{(\omega)}{\nabla}_{[c}|T^{a}{}_{b},|_{d]}+T_{[c}|^{a},{}_{e}T|_{d]}{}^{e},{}_{b}+T_{[c}|_{e},{}^{a}T|_{d]b},{}^{e}$$

$$-T_{[c}|^{a},{}_{e}T^{e}{}_{b},|_{d]}-T_{[d}|_{b},{}^{e}T_{e}{}^{a}|_{c]}-T^{a}{}_{e[c}T_{d]}{}^{e},{}_{b}-T_{b}{}^{e},{}_{[d}T_{c]e},{}^{a}$$

$$+T_{[c}|^{a},{}_{e}T_{b|d]},{}^{e}+T_{e[c}|,{}^{a}T|_{d]}{}^{e},{}_{b}+T^{a}{}_{e},{}_{[c}|T^{e}{}_{b},|_{d]} \tag{A13}$$

$$
\begin{aligned}
2\underset{(\Gamma)}{R}{}_{cd,}{}^{a}{}_{b} = {} & 2\underset{(\omega)}{R}{}_{cd,}{}^{a}{}_{b} + 2\underset{(\omega)}{\nabla}{}_{[c}T_{d]}{}^{a}{}_{,b} + 2\underset{(\omega)}{\nabla}{}_{[c|}T^{a}{}_{b,|d]} - 2\underset{(\omega)}{\nabla}{}_{[c}T_{d]b,}{}^{a} - T_{[c|}{}^{a}{}_{,e}T_{|d]}{}^{e}{}_{,b} \\
& - T_{[c|e,}{}^{a}T_{|d]b,}{}^{e} + T_{[c|}{}^{a}{}_{,e}T^{a}{}_{b,|d]} + T_{[d|b,}{}^{e}T_{e}{}^{a}{}_{|c]} + T^{a}{}_{e,[c}T_{d]}{}^{e}{}_{,b} + T_{b}{}^{e}{}_{,[d}T_{c]e,}{}^{a} \\
& - T_{[c|}{}^{a}{}_{,e}T_{b|d],}{}^{e} - T_{e[c|,}{}^{a}T_{|d]}{}^{e}{}_{,b} - T^{a}{}_{e,[c}T^{e}{}_{b,|d]}.
\end{aligned} \tag{A14}
$$

The symbol $[a|\ldots|b] = a\ldots b - b\ldots a$ stands for antisymmetrisation. The covariant derivatives are defined through the corresponding form relations $D_\omega = e^a \underset{(\omega)}{\nabla}{}_a$ and $D_\Gamma = e^a \underset{(\Gamma)}{\nabla}{}_a$.

The relevant piece of the action may be computed from

$$
\int_{M_{II}} \mathrm{d}^4x\sqrt{-g}\{\underset{(\Gamma)}{R}{}_{ab,cd}\underset{(\Gamma)}{R}{}^{ab,cd} + \underset{(\Gamma)}{\nabla}{}_{[a}K_{b],cd}\underset{(\Gamma)}{\nabla}{}^{[a}K^{b]cd}\}. \tag{A15}
$$

The conditions $\rho = 0$ in component language reads

$$
\tfrac{1}{4}\varepsilon_{cd}{}^{kl}\varepsilon_{ab}{}^{mn}\underset{(\Gamma)}{R}{}_{kl,mn} = \eta\underset{(\Gamma)}{\nabla}{}_{[c}K_{d],ab}. \tag{A16}
$$

References

Arkuszewski W, Kopczynski W and Ponomariev V N 1974 *Ann. Inst. H. Poincaré* **XXIA** 89
Baekler P 1981 *Phys. Lett.* **99B** 329
Benn I M, Dereli T and Tucker R W 1980a *J. Phys. A: Math. Gen.* **13** L359
—— 1980b *Phys. Lett.* **96B** 100
—— 1980c *Double Dual Solutions of Generalised Theories of Gravitation* to be published in *GRG*
Cartan E 1922 *Comptes Rendus* **174** 593
Collins P A and Tucker R W 1976 *Nucl. Phys.* B **112** 150
Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space–Time* (Cambridge: UP)
Howe P S and Tucker R W 1978 *J. Math. Phys.* **19** 869 981
Israel W 1965 *Nuovo Cimento* B **44** 1
Kobayashi S and Nomizu K 1969 *Foundations of Differential Geometry*, vol 2 (New York: Interscience)
Landau L D and Lifshitz E M 1970 *Theory of Elasticity* 2nd English Edn (Oxford: Pergamon) ch 4
Moffat J W and Taylor J G 1980 *Prevention of Gravitational Collapse, University of Toronto preprint*
Misner C W and Sharp D H 1964 *Phys. Rev.* **136** B571
Penrose R 1969 *Riv. Nuov. Cim.* **1** 252
Sakharov A D 1967 *Sov. Phys. Dokl.* **12** 1040
Synge J L 1966 *Relativity: The General Theory* (Amsterdam: North-Holland)
Trautman A 1973 *Nature Phys. Sci.* **242** No 114 pp 7–8